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TOTAL COLORINGS OF DEGENERATE GRAPHS

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A total coloring of a graph G is a coloring of all elements of G, i.e. vertices and edges, such that no two adjacent or incident elements receive the same color. A graph G is s-degenerate for a positive integer s if G can be reduced to a trivial graph by successive removal of vertices with degree $\leq s$. We prove that an s-degenerate graph G has a total coloring with $\Delta+1$ colors if the maximum degree Δ of G is sufficiently large, say $\Delta \geq 4s+3$. Our proof yields an efficient algorithm to find such a total coloring. We also give a linear-time algorithm to find a total coloring of a graph G with the minimum number of colors if G is a partial k-tree, that is, the tree-width of G is bounded by a fixed integer k.

1. Introduction

We deal throughout with a *simple* graph G, which has no loops or multiple edges. A *total coloring* of G is an assignment of colors to its vertices and edges such that no two adjacent vertices or adjacent edges have the same color, and no edge has the same color as one of its endvertices, as illustrated in Fig. 1. The minimum number of colors required for a total coloring of G is called the *total chromatic number* of G, and is denoted by $\chi_{\mathrm{t}}(G)$. The maximum degree, chromatic number and chromatic index (i.e., edge chromatic number) of G are denoted by $\Delta(G)$, $\chi(G)$ and $\chi'(G)$, respectively. It is clear that $\chi_{\mathrm{t}}(G) \geq \Delta(G)+1$, and it is conjectured (the *Total Coloring Conjecture*) that $\chi_{\mathrm{t}}(G) \leq \Delta(G)+2$ for every simple graph G [2,16]. The *total coloring problem* is to find a total coloring of a given graph G with the minimum number $\chi_{\mathrm{t}}(G)$ of colors. Since the problem is NP-hard [14], it is

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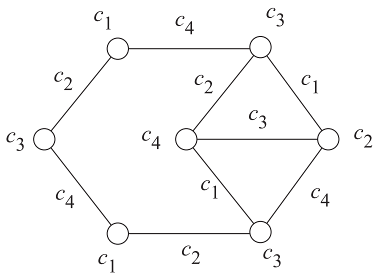

Fig. 1. A total coloring of a graph G with $\chi_t(G)=4$ colors c_1, c_2, c_3 and c_4.

very unlikely that there exists an efficient algorithm to solve the problem for general graphs. However, we will see that there exist efficient algorithms for restricted classes of graphs, specifically for s-degenerate graphs and partial k-trees defined below.

A graph is said to be *s-degenerate* for an integer $s \geq 1$ if it can be reduced to a trivial graph by successive removal of vertices with degree $\leq s$. For example, every planar graph is 5-degenerate. If G is an s-degenerate graph, then it is easy to see that $\chi(G) \leq s+1$, and Vizing [17] proved that $\chi'(G) = \Delta(G)$ if $\Delta(G) \geq 2s$. Thus there is a simple sufficient condition on $\Delta(G)$, i.e. $\Delta(G) \geq 2s$, for the chromatic index $\chi'(G)$ of an s-degenerate graph G to be equal to the trivial lower bound $\Delta(G)$. However, it has not been known until now whether there is a simple sufficient condition on $\Delta(G)$ for the total chromatic number $\chi_t(G)$ to be equal to the trivial lower bound $\Delta(G)+1$.

A graph with bounded tree-width k is called a *partial k-tree*; the formal definition of partial k-trees will be given in Section 2. Any partial k-tree is k-degenerate, but the converse is not always true. Many combinatorial problems can be efficiently solved for partial k-trees with bounded k [1,3, 5]. In particular, both the vertex-coloring problem and the edge-coloring problem can be solved in linear time for partial k-trees [3,18]. However, no efficient algorithm has been known until now for the total coloring problem on partial k-trees. Although the total coloring problem can be solved in polynomial time for partial k-trees by a dynamic programming algorithm, the time complexity $O(n^{1+2^{4(k+1)}})$ is very high [8].

In this paper, we first present a sufficient condition on $\Delta(G)$ for the total chromatic number $\chi_t(G)$ of an s-degenerate graph G to be equal to the trivial lower bound $\Delta(G)+1$: we prove our main theorem that $\chi_t(G) = \Delta(G)+1$ if $\Delta(G) \geq 4s+3$. Our proof immediately yields an efficient algorithm to find a total coloring of G with $\chi_t(G) = \Delta(G)+1$ colors in time $O(sn^2)$ if $\Delta(G) \geq 4s+3$, where n is the number of vertices in G. The complexity can be improved to $O(n \log n)$ in particular if $\Delta(G) \geq 6s+1$ and $s = O(1)$. Hence

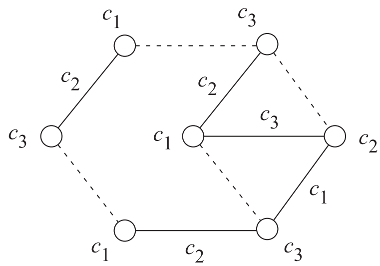

Fig. 2. A partial total coloring of the graph G in Fig. 1 for $U = V$ and $F \subseteq E$ with $\chi_t(G; U, F) = 3$ colors c_1, c_2 and c_3.

the total coloring problem can be solved in time $O(n \log n)$ for a fairly large class of graphs including all planar graphs with sufficiently large maximum degree. We finally show that the total coloring problem can be solved in linear time for partial k-trees G with bounded k. Earlier versions of this paper were presented at [9,10].

2. Preliminaries

In this section we give some basic terminology and definitions, and present key ideas and easy observations.

We denote by $G = (V, E)$ a simple undirected graph with vertex set V and edge set E. Let $n = |V|$ throughout the paper. We denote by vw an edge joining vertices v and w. We denote by $d(v, G)$ the degree of a vertex v in G. For a set $F \subseteq E$, we denote by $G_F = (V, F)$ the spanning subgraph of G induced by the edge set F. A subgraph of G is called a *forest* of G if each connected component is a tree. A forest of G is called a *linear forest* of G if each connected component is a single isolated vertex or a path. For example, $G_F = (V, F)$ is a linear forest of the graph G in Fig. 2 if F consists of the five edges drawn by solid lines.

One of the key ideas in the proof of our main theorem is to introduce a "partial total coloring", which generalizes vertex-, edge- and total colorings. Let C be a set of colors, let U be a subset of V, and let F be a subset of E. Then a *partial total coloring of a graph G for U and F* is a mapping $f: U \cup F \rightarrow C$ such that

(i) $f(v) \neq f(w)$ if $v, w \in U$ and $vw \in E$;
(ii) $f(e) \neq f(e')$ if $e, e' \in F$ and e and e' share a common end; and
(iii) $f(v) \neq f(e)$ if $v \in U$, $e \in F$ and e is incident to v.

The vertices in $\overline{U}=V-U$ and the edges in $\overline{F}=E-F$ are not colored by f. Fig. 2 depicts a partial total coloring of the graph G in Fig. 1 for $U=V$ and F, where $C=\{c_1,c_2,c_3\}$ and all edges in F are drawn as solid lines.

The minimum number of colors required for a partial total coloring of G for U and F is called the *partial total chromatic number of G for U and F*, and is denoted by $\chi_{\rm t}(G;U,F)$. Then obviously $\Delta(G_F)+1 \le \chi_{\rm t}(G;V,F)$. Clearly $\chi_{\rm t}(G) = \chi_{\rm t}(G;V,E)$, $\chi(G) = \chi_{\rm t}(G;V,\emptyset)$, and $\chi'(G) = \chi_{\rm t}(G;\emptyset,E)$. Hence a partial total coloring is an extension of three colorings: a total coloring, a vertex-coloring, and an edge-coloring.

Another idea is a "superimposing" of colorings. Suppose that g is a partial total coloring of a graph $G=(V,E)$ for $U=V$ and $F\subseteq E$, h is an edge-coloring of $G_{\overline{F}}=(V,E-F)$, and g and h use no common color. Then, superimposing g on h, one can obtain a total coloring f of G, and hence

(1) $$\chi_{\rm t}(G) \le \chi_{\rm t}(G;V,F) + \chi'(G_{\overline{F}}).$$

The total coloring f constructed from g and h may use more than $\chi_{\rm t}(G)$ colors even if g uses the minimum number $\chi_{\rm t}(G;V,F)$ of colors and h uses the minimum number $\chi'(G_{\overline{F}})$ of colors, because equality does not always hold in (1). For example, for the graph G in Fig. 2, $\chi_{\rm t}(G)=4$, $\chi_{\rm t}(G;V,F)=3$, $\chi'(G_{\overline{F}})=2$, and hence $\chi_{\rm t}(G)<\chi_{\rm t}(G;V,F)+\chi'(G_{\overline{F}})$. However, we will show in Section 3 as the main theorem that if G is an s-degenerate graph and $\Delta(G)\ge 4s+3$ then there is a subset $F\subset E$ such that equality holds in (1) and $\chi_{\rm t}(G;V,F)+\chi'(G_{\overline{F}})=\Delta(G)+1$, that is,

(2) $$\chi_{\rm t}(G) = \chi_{\rm t}(G;V,F) + \chi'(G_{\overline{F}}) = \Delta(G)+1.$$

We will show in Section 4 that one can efficiently find a partial total coloring g of G for V and F with $\chi_{\rm t}(G;V,F)$ colors and an edge-coloring h of $G_{\overline{F}}$ with $\chi'(G_{\overline{F}})$ colors, and hence one can efficiently find a total coloring f of G with $\chi_{\rm t}(G)$ colors simply by superimposing g and h.

We now recursively define a k-tree: a graph $G=(V,E)$ is a *k-tree* if it is a complete graph of k vertices or it has a vertex $v\in V$ of degree k whose neighbors induce a clique of size k and the graph $G-v$ obtained from G by deleting the vertex v and all edges incident to v is again a k-tree. We then define a partial k-tree: a graph is a *partial k-tree* if it is a subgraph of a k-tree [1,3,18]. The graph in Fig. 1 is indeed a partial 3-tree.

By the definition of an s-degenerate graph $G=(V,E)$, there exists a numbering $\varphi : V\to\{1,2,\ldots,n\}$ such that every vertex $v\in V$ has at most s neighbors numbered by φ with integers larger than $\varphi(v)$, that is,

$$|\{x\in V : vx\in E,\ \varphi(v)<\varphi(x)\}| \le s.$$

Such a numbering φ is called an *s-numbering of G*. An s-numbering of an s-degenerate graph G can be found in linear time [12]. One can easily observe that a graph is 1-degenerate if and only if it is a forest, and that any s-degenerate graph G has at most sn edges.

For a vertex v in a graph $G=(V,E)$ and a numbering $\varphi:V\to\{1,2,\ldots,n\}$, we write

$$\begin{aligned} E_\varphi^{\mathrm{fw}}(v,G) &= \{vx\in E : \varphi(v)<\varphi(x)\};\\ E_\varphi^{\mathrm{bw}}(v,G) &= \{vx\in E : \varphi(v)>\varphi(x)\};\\ d_\varphi^{\mathrm{fw}}(v,G) &= |E_\varphi^{\mathrm{fw}}(v,G)|;\ \text{and}\\ d_\varphi^{\mathrm{bw}}(v,G) &= |E_\varphi^{\mathrm{bw}}(v,G)|. \end{aligned}$$

The edges in $E_\varphi^{\mathrm{fw}}(v,G)$ are called the *forward edges of v*, and those in $E_\varphi^{\mathrm{bw}}(v,G)$ the *backward edges of v*. Clearly $d(v,G)=d_\varphi^{\mathrm{fw}}(v,G)+d_\varphi^{\mathrm{bw}}(v,G)$. A graph G is s-degenerate if and only if there is a numbering φ such that $d_\varphi^{\mathrm{fw}}(v,G)\le s$ for every vertex $v\in V$.

An s-degenerate graph has the following three favorable properties on colorings, two of which have been mentioned in the Introduction.

Lemma 2.1. *For any s-degenerate graph G, the following* (a), (b) *and* (c) *hold:*

(a) $\chi(G)\le s+1$;
(b) *if $\Delta(G)\ge 2s$, then $\chi'(G)=\Delta(G)$* [17]; *and*
(c) $\chi_{\mathrm{t}}(G)\le\Delta(G)+s+1$.

Proof. (a) One can greedily color the vertices of G with at most $s+1$ colors by using as s-numbering (coloring the vertices in decreasing order). See for example [7,11,12,15].

(b) This classical result by Vizing is rather unknown [11]. The original proof written in Russian is given in [17], and a proof in English can be found in [19].

(c) Since G is a simple graph, $\chi'(G)\le\Delta(G)+1$ [16]. Given a set of $\Delta(G)+s+1$ colors, one first uses these colors to color the edges of G, and then colors the vertices greedily using an s-numbering as in (a): at the moment of coloring a vertex v, the forbidden colors are the colors of at most Δ edges incident with v and the colors of at most s colored neighbors of v. ∎

3. Main Theorem

In this section we prove the following main theorem.

Theorem 3.1. *If G is an s-degenerate graph and $\Delta(G) \geq 4s+3$, then* $\chi_{\mathrm{t}}(G) = \Delta(G)+1$.

A result by Borodin *et al.* on total *list* colorings implies that $\chi_{\mathrm{t}}(G) = \Delta(G)+1$ if G is an s-degenerate graph and $\Delta(G) \geq 2s^2$ [4]. In an earlier version [9] of this paper, we showed that $\chi_{\mathrm{t}}(G) = \Delta(G)+1$ if G is s-degenerate and $\Delta(G) \geq 4s^2$. Theorem 3.1 is better than these results.

Our proof of Theorem 3.1 is based on an easy observation (Lemma 2.1(a)), Vizing's result (Lemma 2.1(b)), and König's theorem: $\chi'(G) = \Delta(G)$ for every bipartite graph G. We show in the remainder of this section that there is a subset $F \subset E$ satisfying (2). F will be found as a union of $s+1$ edge-disjoint linear forests of G.

We first show in the following lemma that G can be decomposed into $s+1$ edge-disjoint forests.

Lemma 3.2. *If $G = (V, E)$ is an s-degenerate graph, then there exists a partition $\{F_1, F_2, \ldots, F_{s+1}\}$ of E such that, for any index $j \in \{1, 2, \ldots, s+1\}$,*

(a) *G_{F_j} is a forest; and*
(b) *$d(v, G_{F_j}) \geq 3$ if $d(v, G) \geq 4s+3$.*

Proof. Let $G = (V, E)$ be an s-degenerate graph, and let $\varphi : V \to \{1, 2, \ldots, n\}$ be an s-numbering of G.

We first find $F_1, F_2, \ldots, F_{s+1}$. Construct a new graph $\widetilde{G} = (\widetilde{V}, \widetilde{E})$ from G as follows: for each vertex $v \in V$,

- let $t = \lceil d^{\mathrm{bw}}_{\varphi}(v, G)/(s+1) \rceil$, and replace v with $t+1$ copies of it labeled v_{fw}, $v^1_{\mathrm{bw}}, v^2_{\mathrm{bw}}, \ldots, v^t_{\mathrm{bw}}$;
- attach all forward edges in $E^{\mathrm{fw}}_{\varphi}(v, G)$ to the copy v_{fw}; and
- let $\{E^1_{\mathrm{bw}}, E^2_{\mathrm{bw}}, \ldots, E^t_{\mathrm{bw}}\}$ be any partition of the set $E^{\mathrm{bw}}_{\varphi}(v, G)$ of backward edges such that

$$(3) \qquad |E^i_{\mathrm{bw}}| \begin{cases} = s+1 \text{ if } 1 \leq i \leq t-1; \\ \leq s+1 \text{ if } i = t, \end{cases}$$

  and attach all edges in E^i_{bw} to the copy v^i_{bw} for each $i = 1, 2, \ldots, t$. (See Fig. 3.)

Clearly, $\widetilde{G}$ is bipartite. Since φ is an s-numbering of G, $d(v_{\mathrm{fw}}, \widetilde{G}) = d^{\mathrm{fw}}_{\varphi}(v, G) \leq s$ for every vertex $v \in V$. By (3) $d(v^i_{\mathrm{bw}}, \widetilde{G}) \leq s+1$ for every vertex $v \in V$ and every index $i \in \{1, 2, \ldots, t\}$. Thus we have $\Delta(\widetilde{G}) \leq s+1$.

Since $\widetilde{G}$ is bipartite and $\Delta(\widetilde{G}) \leq s+1$, König's theorem implies that $\chi'(\widetilde{G}) = \Delta(\widetilde{G}) \leq s+1$, and hence $\widetilde{G}$ has an edge-coloring $f : \widetilde{E} \to C$ for a set $C = \{c_1, c_2, \ldots, c_{s+1}\}$ of $s+1$ colors. For each color $c_j \in C$, let $\widetilde{F}_j$ be the color

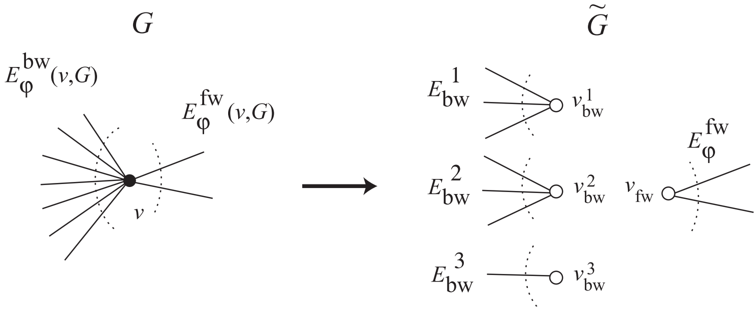

Fig. 3. Transformation from G to $\widetilde{G}$, where $d_\varphi^{\text{fw}}(v,G)=2$, $d_\varphi^{\text{bw}}(v,G)=7$, $s=2$ and $t=3$.

class of c_j, that is, $\widetilde{F}_j=\{e\in\widetilde{E}:f(e)=c_j\}$, and let F_j be the set of edges in E corresponding to $\widetilde{F}_j$. Since f is an edge-coloring of $\widetilde{G}$, $\{\widetilde{F}_1,\widetilde{F}_2,\ldots,\widetilde{F}_{s+1}\}$ is a partition of $\widetilde{E}$, and hence $\{F_1,F_2,\ldots,F_{s+1}\}$ is a partition of E. Thus we have found $F_1,F_2,\ldots,F_{s+1}$.

We then prove that F_j found as above satisfies (a) and (b) for any index $j\in\{1,2,\ldots,s+1\}$.

(a) We shall show that G_{F_j} is 1-degenerate. It suffices to prove that the s-numbering φ of G is indeed a 1-numbering of G_{F_j}, that is, $d_\varphi^{\text{fw}}(v,G_{F_j})\le 1$ for each vertex $v\in V$. By the construction of $\widetilde{G}$, all forward edges of v in G are attached to the copy v_{fw} in $\widetilde{G}$. At most one of them is colored with c_j by f since f is an edge-coloring of $\widetilde{G}$. Thus $\widetilde{F}_j$ contains at most one edge incident to v_{fw}, and hence $d_\varphi^{\text{fw}}(v,G_{F_j})\le 1$.

(b) Let v be any vertex in V with $d(v,G)\ge 4s+3$.

We first claim that $d(v_{\text{bw}}^i,\widetilde{G})=s+1$ for each $i\in\{1,2,3\}$. By the construction of $\widetilde{G}$, $d(v_{\text{bw}}^i,\widetilde{G})=s+1$ if $i\le\left\lfloor d_\varphi^{\text{bw}}(v,G)/(s+1)\right\rfloor$. It therefore suffices to prove that $\left\lfloor d_\varphi^{\text{bw}}(v,G)/(s+1)\right\rfloor\ge 3$. Clearly, $d_\varphi^{\text{fw}}(v,G)\le s$ and $d(v,G)=d_\varphi^{\text{fw}}(v,G)+d_\varphi^{\text{bw}}(v,G)$. Hence

$$(4)\qquad \left\lfloor\frac{d_\varphi^{\text{bw}}(v,G)}{s+1}\right\rfloor=\left\lfloor\frac{d(v,G)-d_\varphi^{\text{fw}}(v,G)}{s+1}\right\rfloor\ge\left\lfloor\frac{4s+3-s}{s+1}\right\rfloor=3.$$

Thus we have proved the claim.

We then prove that $d(v,G_{F_j})\ge 3$. The edge-coloring f of $\widetilde{G}$ uses exactly $s+1$ colors in C, and $d(v_{\text{bw}}^i,\widetilde{G})=s+1$ for each $i\in\{1,2,3\}$. Therefore exactly one of the edges incident to v_{bw}^i in $\widetilde{G}$ is colored with $c_j\in C$ by f. We thus have $d(v,G_{F_j})\ge 3$. ∎

One can construct a linear forest from a forest as in the following lemma.

Lemma 3.3. *Let $T=(V,F)$ be a forest, let $S=\{v\in V:d(v,T)\geq 3\}$, and let U be any subset of S. Then T has a linear forest $T_L=(V,L)$, $L\subseteq F$, such that every vertex in U is an end of a path in T_L, and every vertex in $S-U$ is an interior vertex of a path in T_L, that is,*

$$d(v,T_L)=\begin{cases}1 \text{ if } v\in U;\\ 2 \text{ if } v\in S-U.\end{cases} \qquad (5)$$

Furthermore L can be found in linear time.

Proof. We may assume that the forest T is connected, that is, T is a tree; otherwise, apply the following procedure to each of the connected components (i.e. trees), and let L be the union of their solutions. Choose any vertex r in T, and regard T as a tree rooted at r. Let h be the height of T. Then one can find a subset L of F by the following procedure:

```
procedure Linear-Forest(T, S, U)
begin
let L := ∅; {initialization}
{for each vertex v ∈ S from the root r to leaves,
      choose one or two edges incident to v for L}
for d := 0 to h do
  for each vertex v ∈ S with depth d do
    if v ∈ U then
      if either v is the root r or the edge joining v and its parent
      is not contained in L then
        choose any child v' ∈ V of v in T and let L := L ∪ {vv'}
      fi
    else {v ∈ S − U}
      if either v is the root r or the edge joining v and its parent
      is not contained in L then
        choose any two children v', v'' ∈ V of v in T
        and let L := L ∪ {vv', vv''}
      else  {the edge joining v and its parent is contained in L}
        choose any child v' ∈ V of v and let L := L ∪ {vv'};
      fi
    fi
```

return L;

end.

Clearly, the procedure correctly finds L satisfying (5), and takes linear time. ∎

By Lemmas 3.2 and 3.3 one can find $s+1$ linear forests $G_{L_1}, G_{L_2}, \ldots, G_{L_{s+1}}$ of G as in the following lemma.

Lemma 3.4. *If $G = (V, E)$ is an s-degenerate graph and $\Delta(G) \geq 4s+3$, then for any partition $\{U_1, U_2, \ldots, U_{s+1}\}$ of V there exist mutually disjoint subsets $L_1, L_2, \ldots, L_{s+1}$ of E such that*

(a) *for each $j \in \{1, 2, \ldots, s+1\}$, G_{L_j} is a linear forest, and*

$$d(v, G_{L_j}) \leq \begin{cases} 1 \text{ if } v \in U_j; \\ 2 \text{ if } v \in V - U_j; \end{cases} \tag{6}$$

(b) *$\Delta(G_F) = 2s+1$, where $F = L_1 \cup L_2 \cup \cdots \cup L_{s+1}$; and*
(c) *$\Delta(G_F) + \Delta(G_{\overline{F}}) = \Delta(G)$, where $\overline{F} = E - F$.*

Proof. Let $G = (V, E)$ be an s-degenerate graph, and let $\Delta(G) \geq 4s+3$. We find $L_1, L_2, \ldots, L_{s+1}$ as follows.

We first construct a new graph $G^* = (V^*, E^*)$ from G as follows: for each vertex $v \in V$ with $d(v, G) < 4s+3$, add $(4s+3) - d(v, G)$ dummy vertices and join each of them with v by a dummy edge. Clearly

$$d(v, G^*) = \begin{cases} d(v, G) & \text{if } v \in V \text{ and } d(v, G) \geq 4s+3; \\ 4s+3 & \text{if } v \in V \text{ and } d(v, G) < 4s+3; \text{ and} \\ 1 & \text{if } v \in V^* - V, \end{cases} \tag{7}$$

and hence

$$V = \{v \in V^* : d(v, G^*) \geq 4s+3\}. \tag{8}$$

Since $\Delta(G) \geq 4s+3$,

$$\Delta(G^*) = \Delta(G). \tag{9}$$

We then find $s+1$ forests of G^*. Since G is s-degenerate, G^* is also s-degenerate. Therefore, applying Lemma 3.2 to G^*, one can know that there exists a partition $\{F_1, F_2, \ldots, F_{s+1}\}$ of E^* such that, for any index $j \in \{1, 2, \ldots, s+1\}$,

(i) $G^*_{F_j}$ is a forest; and
(ii) $d(v, G^*_{F_j}) \geq 3$ if $d(v, G^*) \geq 4s+3$.

By (ii), (7) and (8) we have $V = \{v \in V^* : d(v, G^*_{F_j}) \geq 3\}$.

We then find $s+1$ linear forests of G^*. Let $\{U_1, U_2, \ldots, U_{s+1}\}$ be any partition of V. For each $j \in \{1, 2, \ldots, s+1\}$, apply Lemma 3.3 to $T = G^*_{F_j}$, $S = V = \{v \in V^* : d(v, G^*_{F_j}) \geq 3\}$ and $U = U_j \subseteq V$, then one can know that the forest $G^*_{F_j}$ has a linear forest $G^*_{L^*_j} = (V^*, L^*_j)$ such that

$$(10) \qquad d(v, G^*_{L^*_j}) = \begin{cases} 1 \text{ if } v \in U_j; \\ 2 \text{ if } v \in V - U_j. \end{cases}$$

Since $L^*_j \subseteq F_j$, $1 \leq j \leq s+1$, and $F_1, F_2, \ldots, F_{s+1}$ are mutually disjoint, $L^*_1, L^*_2, \ldots, L^*_{s+1}$ are also mutually disjoint.

We then find $L_1, L_2, \ldots, L_{s+1}$ from $L^*_1, L^*_2, \ldots, L^*_{s+1}$; for each $j \in \{1, 2, \ldots, s+1\}$, let L_j be the set of all non-dummy edges in L^*_j, that is, $L_j = L^*_j \cap E$. Then $L_j \subseteq L^*_j$. Furthermore, one can easily observe that

$$(11) \qquad d(v, G_{L_j}) \begin{cases} \leq d(v, G^*_{L^*_j}) \text{ if } v \in V; \\ = d(v, G^*_{L^*_j}) \text{ if } v \in V \text{ and } d(v, G) \geq 4s+3. \end{cases}$$

Since $L^*_1, L^*_2, \ldots, L^*_{s+1}$ are mutually disjoint, $L_1, L_2, \ldots, L_{s+1}$ are also mutually disjoint. Thus we have found $L_1, L_2, \ldots, L_{s+1}$.

We shall prove that $L_1, L_2, \ldots, L_{s+1}$ and $F = L_1 \cup L_2 \cup \cdots \cup L_{s+1}$ satisfy (a)–(c).

(a) Since $L_j \subseteq L^*_j$ and $G^*_{L^*_j}$ is a linear forest of G^*, G_{L_j} is a linear forest of G. Let v be any vertex in V. If $v \in U_j$, then by (10) and (11) we have $d(v, G_{L_j}) \leq d(v, G^*_{L^*_j}) = 1$. Similarly, if $v \in V - U_j$, then we have $d(v, G_{L_j}) \leq d(v, G^*_{L^*_j}) = 2$.

(b) Let $F^* = L^*_1 \cup L^*_2 \cup \cdots \cup L^*_{s+1}$, so that $F \subseteq F^*$ since $L_j \subseteq L^*_j$ for each j. Let $v \in V$. Since $v \in U_j$ for exactly one j, (10) gives

$$(12) \qquad d(v, G^*_{F^*}) = \sum_{i=1}^{s+1} d(v, G^*_{L^*_i}) = 2s+1.$$

However, by (11), we have $d(v, G_F) \leq d(v, G^*_{F^*})$, with equality if $d(v, G) \geq 4s+3$. Since $\Delta(G) \geq 4s+3$ by hypothesis, it follows that $\Delta(G_F) = 2s+1$.

(c) Clearly $\Delta(G_F) + \Delta(G_{\overline{F}}) \geq \Delta(G)$. We shall therefore prove that $\Delta(G_F) + \Delta(G_{\overline{F}}) \leq \Delta(G)$, that is, $\Delta(G_{\overline{F}}) \leq \Delta(G) - \Delta(G_F)$. Since $E \subseteq E^*$ and $F = F^* \cap E$,

$$(13) \qquad E - F \subseteq E^* - F^*.$$

For each vertex $v\in V$, by (b) above and (9), (12) and (13) we have

$$\begin{aligned} d(v,G_{\overline{F}}) &= d(v,G_{E-F}) \\ &\le d(v,G^*_{E^*-F^*}) \\ &= d(v,G^*) - d(v,G^*_{F^*}) \\ &\le \Delta(G^*) - (2s+1) \\ &= \Delta(G) - \Delta(G_F). \end{aligned}$$

Thus we have $\Delta(G_{\overline{F}})\le\Delta(G)-\Delta(G_F)$. ∎

By Lemma 2.1(a) any s-degenerate graph G has a vertex-coloring with $s+1$ colors. Choose the set of color classes as the partition $\{U_1,U_2,\dots,U_{s+1}\}$ in Lemma 3.4. Then there is a subset $F\subset E$ satisfying (2), as shown in the following theorem. The key idea of the proof is to construct a partial total coloring of G for $U=V$ and F from partial total colorings of G for U_j and L_j, $1\le j\le s+1$, by superimposing them.

Theorem 3.5. *If $G=(V,E)$ is an s-degenerate graph and $\Delta(G)\ge 4s+3$, then there exists a subset F of E such that*

(a) $\chi_t(G;V,F)=\Delta(G_F)+1$;
(b) $\chi'(G_{\overline{F}})=\Delta(G_{\overline{F}})$, *where* $\overline{F}=E-F$;
(c) $\Delta(G_F)+\Delta(G_{\overline{F}})=\Delta(G)$; *and*
(d) $\chi_t(G)=\chi_t(G;V,F)+\chi'(G_{\overline{F}})=\Delta(G)+1$.

Proof. Let $G=(V,E)$ be an s-degenerate graph, and let $\Delta(G)\ge 4s+3$. Since G is s-degenerate, by Lemma 2.1(a) $\chi(G)\le s+1$ and hence G has a vertex-coloring $f:V\to C$ for a set $C=\{c_1,c_2,\dots,c_{s+1}\}$ of $s+1$ colors. For each color $c_j\in C$, let U_j be the color class of c_j, that is, $U_j=\{v\in V: f(v)=c_j\}$. Then $\{U_1,U_2,\dots,U_{s+1}\}$ is a partition of V and each of $U_1,U_2,\dots,U_{s+1}$ is an independent set of G. By Lemma 3.4 for the partition $\{U_1,U_2,\dots,U_{s+1}\}$ there exist mutually disjoint subsets $L_1,L_2,\dots,L_{s+1}$ of E satisfying the conditions (a)–(c) in Lemma 3.4. Since the condition (c) in Theorem 3.5 is the same as (c) in Lemma 3.4, we shall show that $F=L_1\cup L_2\cup\cdots\cup L_{s+1}$ satisfies the conditions (a), (b) and (d) in Theorem 3.5.

(a) Clearly $\chi_t(G;V,F)\ge\Delta(G_F)+1$. Therefore it suffices to prove that $\chi_t(G;V,F)\le\Delta(G_F)+1$. Since $\Delta(G_F)=2s+1$ by Lemma 3.4(b), we wish to prove that $\chi_t(G;V,F)\le 2s+2$.

For each $j\in\{1,2,\dots,s+1\}$, we first construct a partial total coloring g_j of G for U_j and L_j using the set $C_j=\{2j-1,2j\}$ of two colors. Since G_{L_j} is a linear forest, its edges can be colored with the two colors in C_j. By (6), each vertex $v\in U_j$ can now be given a color from C_j that is not used on an edge incident with v, and since U_j is an independent set, this gives a partial total coloring as required. Since the sets C_j are disjoint, the colorings $g_1, g_2, \ldots, g_{s+1}$ can be superimposed to give a partial total coloring of G for V and F using $2s+2$ colors. Therefore $\chi_t(G;V,F) \le 2s+2$, and this completes the proof that $\chi_t(G;V,F) = \Delta(G_F)+1$.

(b) Since G is s-degenerate, the subgraph $G_{\overline{F}}$ of G is also s-degenerate. Since $\Delta(G) \ge 4s+3$, by the conditions (b) and (c) in Lemma 3.4 we have $\Delta(G_{\overline{F}}) = \Delta(G) - \Delta(G_F) \ge (4s+3) - (2s+1) = 2s+2 > 2s$. Therefore, by Lemma 2.1(b) we have $\chi'(G_{\overline{F}}) = \Delta(G_{\overline{F}})$.

(d) Clearly $\Delta(G)+1 \le \chi_t(G)$, and $\chi_t(G) \le \chi_t(G;V,F) + \chi'(G_{\overline{F}})$ by (1). However (a), (b) and (c) show that $\chi_t(G;V,F) + \chi'(G_{\overline{F}}) = \Delta(G_F)+1+\Delta(G_{\overline{F}}) = \Delta(G)+1$, and so equality holds throughout. ∎

Theorem 3.5(d) implies Theorem 3.1.

Remark. It is not known whether Vizing's bound $\Delta(G) \ge 2s$ in Lemma 2.1(b) is tight or not. Similarly, we do not know whether our bound $\Delta(G) \ge 4s+3$ in Theorem 3.1 is tight or not. However, there is a 2-degenerate graph G such that $\Delta(G) = 2s-1 = 3$ but $\chi_t(G) = 5 \ne \Delta(G)+1$, for example, a graph obtained from the complete bipartite graph $K_{3,3}$ by deleting an edge. The bound $\Delta(G) \ge 4s+3$ is crucial to the proof of (4). Furthermore, Lemma 3.3 cannot be improved: if $S = \{v \in V : d(v,T) \ge 2\}$, then T does not always have a linear forest T_L satisfying (5). Thus it would not be straightforward to improve the bound $\Delta(G) \ge 4s+3$.

4. Algorithms

From the proofs of Lemmas 3.2–3.4 and Theorem 3.5, one can know that the following algorithm correctly finds a total coloring of an s-degenerate graph $G=(V,E)$ with $\Delta(G)+1$ colors if $\Delta(G) \ge 4s+3$.

Total-Coloring Algorithm.

Step 1. Find a vertex-coloring of a given s-degenerate graph G with $s+1$ colors, and let $\{U_1, U_2, \ldots, U_{s+1}\}$ be the set of color classes.

Step 2. Construct a graph $G^* = (V^*, E^*)$ from G by adding dummy vertices and edges, as in the proof of Lemma 3.4.

Step 3. Construct a bipartite graph $\widetilde{G^*} = (\widetilde{V^*}, \widetilde{E^*})$ from G^* by splitting each vertex in G^*, as illustrated in Fig. 3. Note that $\Delta(\widetilde{G^*}) \le s+1$.

Step 4. Find an edge-coloring of the bipartite graph $\widetilde{G^*}$ with $s+1$ colors, and let $\{\widetilde{F_1^*}, \widetilde{F_2^*}, \ldots, \widetilde{F_{s+1}^*}\}$ be the set of color classes. Let $\{F_1, F_2, \ldots, F_{s+1}\}$ be the partition of E^* corresponding to $\{\widetilde{F_1^*}, \widetilde{F_2^*}, \ldots, \widetilde{F_{s+1}^*}\}$, where $G^*_{F_j}$, $1 \le j \le s+1$, is a forest of G^*.

Step 5. From each forest $G^*_{F_j}$ of G^*, $1 \le j \le s+1$, find a linear forest $G^*_{L^*_j}$ of G^* such that

$$d(v, G^*_{L^*_j}) = \begin{cases} 1 \text{ if } v \in U_j; \\ 2 \text{ if } v \in V - U_j, \end{cases}$$

using the procedure Linear-Forest in the proof of Lemma 3.3, where U_j is a color class found in Step 1.

Step 6. From each linear forest $G^*_{L^*_j}$ of G^*, obtain a linear forest G_{L_j} of G such that

$$d(v, G_{L_j}) \le \begin{cases} 1 \text{ if } v \in U_j; \\ 2 \text{ if } v \in V - U_j, \end{cases}$$

by deleting all dummy vertices and edges as in the proof of Lemma 3.4.

Step 7. For each j, find a partial total coloring g_j of G for U_j and L_j using two colors as in the proof of Theorem 3.5.

Step 8. Superimposing $g_1, g_2, \ldots, g_{s+1}$, obtain a partial total coloring g of G for V and $F = L_1 \cup L_2 \cup \cdots \cup L_{s+1}$ with $\Delta(G_F)+1$ colors, as in the proof of Theorem 3.5.

Step 9. Find an edge-coloring h of $G_{\overline{F}}$ with $\Delta(G_{\overline{F}})$ colors.

Step 10. Superimposing g and h, obtain a total coloring of G with $\Delta(G)+1$ colors.

We then show that all steps can be done in time $O(sn^2)$, using an algorithm for edge-coloring bipartite graphs [6] and an algorithm for edge-coloring s-degenerate graphs [18,19].

One can easily find the vertex-coloring of G in time $O(sn)$ by a simple greedy algorithm based on an s-numbering of G [7,11,12,15]. Note that G has at most sn edges. Thus Step 1 can be done in time $O(sn)$.

By the construction of the graph G^*, we have $|V^* - V| = |E^* - E| \le (4s+3)n$ and hence

$$\begin{aligned} |V^*| &\le n + (4s+3)n = 4(s+1)n, \text{ and} \\ |E^*| &\le sn + (4s+3)n = (5s+3)n. \end{aligned}$$

Thus one can construct the graph G^* in time $O(sn)$, and hence Step 2 can be done in time $O(sn)$.

Clearly

$$\begin{aligned} |\widetilde{E^*}| &= |E^*| \le (5s+3)n, \text{ and} \\ |\widetilde{V^*}| &\le 2|\widetilde{E^*}| \le 2(5s+3)n. \end{aligned}$$

Therefore one can construct $\widetilde{G^*}$ from G^* in time $O(sn)$. Thus Step 3 can be done in time $O(sn)$.

Since $\widetilde{G^*}$ is bipartite and $\Delta(\widetilde{G^*}) \le s+1$, one can find the edge-coloring of $\widetilde{G^*}$ in time $O(|\widetilde{E^*}| \log \Delta(\widetilde{G^*})) = O(sn \log s)$ by the algorithm in [6]. Note that $s \le n$. Thus Step 4 can be done in time $O(sn \log s)$.

By Lemma 3.2, for each forest $G^*_{F_j}$, $1 \le j \le s+1$, one can find the linear forest $G^*_{L^*_j}$ in time $O(|V^*|)=O(sn)$. Therefore the $s+1$ linear forests $G^*_{L^*_1}, G^*_{L^*_2}, \dots, G^*_{L^*_{s+1}}$ can be found in time $O((s+1)sn)=O(s^2n)$. Thus Step 5 can be done in time $O(s^2n)$.

From each linear forest $G^*_{L^*_j}$, $1 \le j \le s+1$, one can obtain the linear forest G_{L_j} in time $O(|L^*_j|)=O(sn)$ simply by deleting dummy vertices and edges. One can therefore obtain the $s+1$ linear forests $G_{L_1}, G_{L_2}, \dots, G_{L_{s+1}}$ in time $O((s+1)sn)=O(s^2n)$. Thus Step 6 can be done in time $O(s^2n)$.

For each j, $1 \le j \le s+1$, one can easily find a partial total coloring g_j in time $O(n)$. Therefore Step 7 can be done in time $O(sn)$.

Superimposing $g_1, g_2, \dots, g_{s+1}$, one can obtain the partial total coloring g of G for V and F in time $O(sn)$. Thus Step 8 can be done in time $O(sn)$.

Since $G_{\overline{F}}$ is s-degenerate, one can find the edge-coloring h of $G_{\overline{F}}$ in time $O(sn^2)$ [13], [18, p. 604], [19, p. 8]. Therefore Step 9 can be done in time $O(sn^2)$.

Superimposing g and h, one can obtain a total coloring of G with $\chi_t(G)=\Delta(G)+1$ colors in time $O(sn)$. Thus Step 10 can be done in time $O(sn)$.

Thus all Steps 1–10 above can be done in time $O(sn^2)$, and hence we have the following theorem.

Theorem 4.1. *A total coloring of an s-degenerate graph G using $\chi_t(G)=\Delta(G)+1$ colors can be found in time $O(sn^2)$ if $\Delta(G) \ge 4s+3$.*

The complexity $O(sn^2)$ can be improved if either $\Delta(G) \ge 6s+1$ and $s=O(1)$ or G is a partial k-tree, as in the following two theorems.

Theorem 4.2. *A total coloring of an s-degenerate graph G using $\chi_t(G)=\Delta(G)+1$ colors can be found in time $O(n \log n)$ if $\Delta(G) \ge 6s+1$ and $s=O(1)$.*

Proof. Let $\Delta(G) \ge 6s+1$ and $s=O(1)$. Since $s=O(1)$, all steps except Step 9 can be done in time $O(n)$. It therefore suffices to show that Step 9 can be done in time $O(n \log n)$.

Since $\Delta(G_F)=2s+1$ and $\Delta(G_F)+\Delta(G_{\overline{F}})=\Delta(G)$ by Lemma 3.4, we have $\Delta(G_{\overline{F}})=\Delta(G)-\Delta(G_F) \ge (6s+1)-(2s+1)=4s$. Therefore the edge-coloring h of the s-degenerate graph $G_{\overline{F}}$ can be found in time $O(n \log n)$ [19, Corollary 2(c)]. Thus Step 9 can be done in time $O(n \log n)$. ∎

Theorem 4.3. *The total coloring problem can be solved in linear time for partial k-trees with bounded k.*

Proof. Let $G=(V,E)$ be a partial k-tree, and let $k=O(1)$. We shall show that a total coloring of G with $\chi_t(G)$ colors can be found in linear time.

We first consider the case where $\Delta(G)$ is small, say $\Delta(G)<4k+3$. Since G is a partial k-tree, G is k-degenerate. Therefore by Lemma 2.1(c) we have $\chi_t(G)\le\Delta(G)+k+1<5k+4=O(1)$. A dynamic programming algorithm in [8] finds a total coloring of a partial k-tree G with $\chi_t=\chi_t(G)$ colors in time $O(n\chi_t^{2^{4(k+1)}})$. Since $\chi_t(G)=O(1)$ and $k=O(1)$, the algorithm takes time $O(n)$ for this case.

We next consider the case where $\Delta(G)$ is large, say $\Delta(G)\ge 4k+3$. Since $s=k=O(1)$, all steps except Step 9 can be done in linear time. The algorithm in [18] finds an edge-coloring of a partial k-tree G with $\chi'(G)$ colors in linear time. Since the subgraph $G_{\overline{F}}$ of G is also a partial k-tree, the algorithm finds an edge-coloring of $G_{\overline{F}}$ in linear time. Thus Step 9 can be done in linear time. ∎

5. Conclusion

In this paper we prove that $\chi_t(G)=\Delta(G)+1$ for an s-degenerate graph G if $\Delta(G)\ge 4s+3$. Our proof immediately yields an algorithm to find a total coloring of G with $\chi_t(G)=\Delta(G)+1$ colors in time $O(sn^2)$. The complexity can be improved either to $O(n\log n)$ if $\Delta(G)\ge 6s+1$ and $s=O(1)$, or to $O(n)$ if G is a partial k-tree. Thus the total coloring problem can be solved in time $O(n\log n)$ or $O(n)$ for a fairly large class of graphs including all planar graphs with sufficently large maximum degree. One can similarly show that a total coloring with $\Delta(G)+1$ colors can be efficiently found for various classes of graphs G with large $\Delta(G)$ having fixed genus, arboricity or thickness, and show that, for any minor-closed class of graphs, the Total Coloring Conjecure holds for graphs in that class with large enough maximum degrees. We hope in future to extend our results on total colorings to total list colorings [4].

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